Micro Prelim Tutoring Session 1

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Fall 2015 Part 1: Question 2

A consumer with utility function $u(x_1, x_2) = x_1 \cdot x_2$ faces budget constraint $2x_1 + 2x_2 = 8$.

a) The price of good 1 is reduced from 2 to 1. Determine the welfare effect of this price reduction. Use both equivalent and compensating variation to do this. Explain why these two measures of the welfare effect of the price change differ from one another.

The generic utility maximization problem under prices (p_1, p_2) and wealth w becomes:
max x_1x_2 st $p_1x_1 + p_2x_2 = w$

Let's set up the Lagrangian
$\mathcal{L} = x_1x_2 - \lambda(p_1x_1 + p_2x_2 - w)$

FOC:
$\frac{\partial \mathcal{L}}{\partial x_1} : x_2 - \lambda p_1 = 0$
$\lambda = \frac{x_2}{p_1}$

$\frac{\partial \mathcal{L}}{\partial x_2} : x_1 - \lambda p_2 = 0$
$\lambda = \frac{x_1}{p_2}$

So, $\frac{x_2}{p_1} = \frac{x_1}{p_2}$
$p_2x_2 = p_1x_1$

From the budget line: $p_1x_1 + p_2x_2 = w$
$w = 2p_1x_1$ (Substituting in $p_2x_2 = p_1x_1$)

We can rewrite x_1 as the demand for x_1
$w = 2p_1 \cdot x_1(p, w)$

So, the demand function of x_1 is:
$x_1(p, w) = \frac{w}{2p_1}$

Since $p_2x_2 = p_1x_1$
$p_2x_2(p, w) = p_1\frac{w}{2p_1}$

So, the demand function of x_2 is:
$x_2(p, w) = \frac{w}{2p_2}$

Let's evaluate the demands and utilty under the original prices (2,2), new prices (1,2), and wealth 8
$x_1((2, 2), 8) = 2$
$x_2((2, 2), 8) = 2$
$u(2, 2) = 4$

$x_1((1,2),8) = 4$
$x_2((1,2),8) = 2$
$u(2,2) = 8$

Since we have the two demand functions, we can find the expenditure function through the indirect utility function and dual problem.
The indirect utility function ($V(p,w)$) is the utility function evaluated at the demand functions. So,
$V(p,w) = x_1(p,w) \cdot x_2(p,w)$
$V(p,w) = \frac{w}{2p_1} \cdot \frac{w}{2p_2}$
$V(p,w) = \frac{w^2}{4p_1p_2}$

Now by the dual problem:
$V(p, e(p,u)) = \overline{u}$
$\frac{e(p,u)^2}{4p_1p_2} = \overline{u}$

Now solve for $e(p,u)$
$e(p,u) = 2\sqrt{\overline{u}p_1p_2}$

Now we can calculate EV and CV
EV=$e(p^o, u') - w$
EV=$e((2,2),8) - 8$
EV=$2\sqrt{8 \cdot 2 \cdot 2} - 8$
EV=$8\sqrt{2} - 8$

EV=$w - e(p', u^o)$
EV=$8 - e((1,2),4)$
EV=$8 - 2\sqrt{4 \cdot 1 \cdot 2}$
EV=$8 - 4\sqrt{2}$

EV and CV measure two different things:
EV) Measures how much of a wealth change would be needed before the price change for the agent to be just as happy as after the price change
CV) Measures how much of a wealth change would be needed after the price change to make the agent just as happy as before the price change

b) Instead of a price change, assume the quantity chosen for good 1 must be less than or equal to 1. Discuss how you would measure the welfare effect of this restriction. Can equivalent and compensating variation be used, and are they different? Compute the welfare effect of this quantity restriction.

$p_1 = p_2 = 2$ and $w = 8$
Based on the demand function found in part (a), the optimal amount of x_1 would be 2. But since we are now limiting the amount to less than or equal to 1, the agent would choose $x_1 = 1$ to maximize utility.
Based on the budget line $2x_1 + 2x_2 = 8$, if $x_1 = 1$, then the agent spends the rest of the income on x_2 which results in $x_2 = 3$
Now the utility from these amounts becomes $u(1,3) = 1 \cdot 3 = 3$

We can still calculate EV since we have the original prices and a new utility:
EV=$e((2,2),3) - w$

EV=$2\sqrt{3 \cdot 2 \cdot 2} - 8$
EV=$4\sqrt{3} - 8$

Calculating CV is a little trickier. We know the original utility under the optimal amounts (2,2) was 4. So we have to figure out what x_2 must equal with $x_1 = 1$ so that we have a utility of 4. So, $u(1, x_2) = x_2 = 4$, $x_2 = 4$.
In order to afford the input amounts (1,4), the agent needs $2 \cdot 1 + 2 \cdot 4 = 10$. So,
CV=8-10
CV=-2

c) The equivalent and compensating variation measures used in part a) are equal to the area under the Hicksian demand function for good one between the two price levels. Determine the Hicksian demand function for this consumer and demonstrate this fact.

We can use the dual problem once again:
$h_1(p, u) = x_1(p, e(p, u))$

From part (a), $x_1(p, w) = \frac{w}{2p_1}$ and $e(p, u) = 2\sqrt{\overline{u}p_1p_2}$
$h_1(p, u) = \frac{e(p,u)}{2p_1}$
$h_1(p, u) = \frac{2\sqrt{\overline{u}p_1p_2}}{2p_1}$
$h_1(p, u) = \sqrt{\frac{\overline{u}p_2}{p_1}}$

So, we can now evalutate under $(p_1, 2)$ $u = 4$ and $u = 8$
$h_1((p_1, 2), 4) = \sqrt{\frac{4 \cdot 2}{p_1}}$
$h_1((p_1, 2), 4) = 2\sqrt{\frac{2}{p_1}}$

$h_1((p_1, 2), 8) = \sqrt{\frac{8 \cdot 2}{p_1}}$
$h_1((p_1, 2), 8) = \frac{4}{\sqrt{p_1}}$

Now we need to show the area under these functions is equal EV=$8\sqrt{2} - 8$ and CV=$8 - 4\sqrt{2}$

EV= $\int_1^2 \frac{4}{\sqrt{p_1}} dp_i$
EV=$4[2\sqrt{p_1}|_1^2]$
EV=$4[2\sqrt{2} - 2]$
EV=$8\sqrt{2} - 8\checkmark$

CV= $\int_1^2 2\sqrt{\frac{2}{p_1}} dp_i$
CV=$2\sqrt{2}[2\sqrt{p_1}|_1^2]$
CV=$2\sqrt{2}[2\sqrt{2} - 2]$
CV=$8 - 4\sqrt{2}\checkmark$

Spring 2015 Part 2: Question 2

Consider the following candidate for a conditional cost function

$$c(w, q) = w_1^a q - 25\frac{w_1^{2a}}{w_2^b},$$

where $w = (w_1, w_2)$ are input prices and q is the output level. Assume that q is sufficiently large so that nonnegativity constraints are satisied.

(i) Find restrictions on parameters a and b so that function $c(\cdot)$ is a well-behaved conditional cost function.

For the function to be well behaved, it must be hom(1) in prices
That is, $C(\alpha w, q) = \alpha C(w, q)$
Translating it into our cost function:
$\alpha c(w, q) = \alpha(w_1^a q - 25\frac{w_1^{2a}}{w_2^b})$
$c(\alpha w, q) = (\alpha w_1)^a q - 25\frac{(\alpha w_1)^{2a}}{(\alpha w_b)^2}$
So we need:
$\alpha(w_1^a q - 25\frac{w_1^{2a}}{w_b^2}) = (\alpha w_1)^a q - 25\frac{(\alpha w_1)^{2a}}{(\alpha w_2)^b}$

Let's distribute the α in $\alpha c(w, q)$:
$\alpha c(w, q) = \alpha w_1^a q - \alpha 25\frac{w_1^{2a}}{w_2^b}$

We need $\alpha w_1^a q = (\alpha w_1)^a q$
The only way this is true is if $a = 1$

If $a = 1$, then the second term in $\alpha c(w, q)$ becomes:
$\alpha 25\frac{w_1^2}{w_2^b}$
We need this term to equal $25\frac{(\alpha w_1)^2}{(\alpha w_2)^b}$
$\alpha 25\frac{w_1^2}{w_2^b} = 25\frac{(\alpha w_1)^2}{(\alpha w_2)^b}$
$\frac{w_1^2}{w_2^b} = \frac{\alpha w_1^2}{\alpha^b w_2^b}$
The only way this holds is if $b = 1$

So both a and b must be equal to 1

(ii) Find the underlying production function $q = f(z_1, z_2)$, where z_1 and z_2 are the two inputs.

By Shepard's Lemma, if we have a firm's cost function, then the demand for each input is the derivative of the cost function wrt to each input's price.

Under $a = b = 1$, the cost function becomes: $w_1 q - 25\frac{w_1^2}{w_2}$

$\frac{\partial C(w,q)}{\partial w_1} = z_1$
$\frac{\partial}{\partial w_1}(w_1 q - 25\frac{w_1^2}{w_2}) = z_1$
$q - 50\frac{w_1}{w_2} = z_1$
$q = z_1 + 50\frac{w_1}{w_2}$

$\frac{\partial C(w,q)}{\partial w_2} = z_2$
$\frac{\partial}{\partial w_2}(w_1 q - 25\frac{w_1^2}{w_2}) = z_2$
$25\frac{w_1^2}{w_2^2} = z_2$

We have the two demand functions, now let's rearrange the demand for z_2 so that we can plug into the demand for z_1 and have a function in terms of z_1, z_2

$(5\frac{w_1}{w_2})^2 = z_2$
$\frac{w_1}{w_2} = \frac{\sqrt{z_2}}{5}$
$q = z_1 + 50\frac{\sqrt{z_2}}{5}$
$q = z_1 + 10\sqrt{z_2}$
$f(z_1, z_2) = z_1 + 10\sqrt{z_2}$

(iii) Given the production function found in (ii), suppose $q < 50w_1/w_2$. Find the corresponding conditional cost function.

To find the conditional cost function, we need to set up the firm's cost minimization problem:

min $w_1 z_1 + w_2 z_2$ st $q = z_1 + 10\sqrt{z_2}$ where $q < \frac{50w_1}{w_2}$

Let's set up the Lagranian:
$\mathcal{L} = w_1 z_1 + w_2 z_2 - \lambda(z_1 + 10\sqrt{z_2} - q) - \mu(q - \frac{50w_1}{w_2})$

FOC:
$\frac{\partial \mathcal{L}}{\partial z_1} : w_1 = \lambda$
$\frac{\partial \mathcal{L}}{\partial z_2} : w_2 = \frac{5\lambda}{\sqrt{z_2}}$

Let's substitute in $w_1 = \lambda$ into $w_2 = \frac{5\lambda}{\sqrt{z_2}}$ and solve for $\sqrt{z_2}$
$w_2 = \frac{5w_1}{\sqrt{z_2}}$
$\sqrt{z_2} = \frac{5w_1}{w_2}$

Since $q = z_1 + 10\sqrt{z_2}$, we can plug in $\sqrt{z_2}$
$q = z_1 + \frac{50w_1}{w_2}$

We need $q < \frac{50w_1}{w_2}$ to hold. So any $z_1 > 0$ violates this condition. Since a firm can't use negative amounts of inputs, $z_1 = 0$ and $z_2 = \frac{25w_1^2}{w_2^2}$ (From $\sqrt{z_2} = \frac{5w_1}{w_2}$). Therefore the conditional cost function becomes:
$C(w, q) = w_1 z_1 + w_2 z_2$
$C(w, q) = w_1 \cdot 0 + w_2\frac{25w_1^2}{w_2^2}$
$C(w, q) = \frac{25w_1^2}{w_2}$

Fall 2016 Part 1: Question 3

Consider an economy with m commodities. A bundle of commodities is a point $q \in R^m_+$. An individual has preferences over $R^m_+ \times [0,1]$, where the additional dimension is the fraction of time that the individual spends in leisure. Assume that the individual's preferences can be represented by a concave utility function is $U(q,l)$, where q is the bundle of commodities she consumes and $l \in [0,1]$ is the leisure hours she enjoys. If the individual spends l time in leisure, she spends $1-l$ time employed at work, drawing a wage $w \in R_{++}$ per unit of time devoted to work. Assume the consumer, besides wages, also has capital income k. Her utility function is strictly increasing in both q and l.

(a) Clearly state the consumer's utility maximizing problem. Assuming that leisure is a normal good, discuss the effect of an increase in the wage rate on her leisure demand. In particular, is it possible that an increase in wage rate leads to a reduction in labor supply? Be very precise in your answer. You can either give a descriptive answer or explicitly state and use the first-order conditions for the consumer's problem to present your argument.

Descriptive:
The individual wants to maximize their utility function $U(q,l)$ st $pq \leq w(1-l)+k$
Where p is the price of commodity bundle q

Since leisure is a normal good, then as income increases, quantity demanded should also increase.
Well, if the wage rate increases there are two effects:
1) Substitution effect: Leisure becomes more expensive, so substitutes away towards labor. So, labor increases
2) Income effect: Income increases, so leisure should increase, but that makes labor decrease
If the income effect dominates, then an increase in wage could cause a reduction in labor

(b) Derive and discuss the form of Roy's Identity for this problem.
Roy's identity: $x_i(p,w) = -\frac{\partial v(p,w)}{\partial p_i} / \frac{\partial v(p,w)}{\partial W}$
Where $W = w(1-l)+k$
Since this version of Roy's identity depends on how w and k are changing, we have to discuss three cases:
Case 1: k is constant, w changes. Then Roy's identity tells us how commodity and leisure demand changes with changes in the wage rate.
Case 2: k changes, w is constant. Then Roy's identity tells us how commodity and leisure demand changes with changes to non-labor income
Case 3: k and w change, but W is constant. Then Roy's identity tells us how commodity and leisure demand changes with changes in the wealth composition